

# Desingularization of branch points of minimal disks in $\mathbb{R}^4$

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## Abstract

We deform a minimal disk in  $\mathbb{R}^4$  with a branch point into symplectic minimally immersed disks with only transverse double points.

## 1 Introduction

This paper continues the study of branch points of minimal disks in  $\mathbb{R}^4$  and their knots which was started in [Vi], [S-V1] and [S-V2]. Near the branch point, the disk is symplectic for two different symplectic structures, one for each orientation in  $\mathbb{R}^4$ . For each of these symplectic structures, we show that the branched disk can be deformed into symplectic minimally immersed disks with only transverse double points. If the branched disk is topologically embedded, this can be done without changing the transverse knot type of the boundary knot and the number of the double points of the immersed disks is given by the self-linking number of the transverse knot.

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## 2 Preliminaries

### 2.1 Branch points

Let  $F : \mathbb{D} \longrightarrow \mathbb{R}^4$  be a map. A point  $p \in \mathbb{D}$  is a branch point of  $F$  if we can find a coordinate system  $(x_i)$  around  $F(p)$  in which the map is written as

$$F_1(z) + iF_2(z) = z^N + o(|z|^N) \quad F_3(z) + iF_4(z) = o(|z|^N) \quad (1)$$

where  $F_i(z)$  denotes the  $i$ -th component of  $F(z)$  in the coordinate system  $(x_i)$ .

Here and throughout this paper,  $p$  identifies with 0 and  $F(p)$  is identified with  $(0, \dots, 0)$  in  $\mathbb{R}^4$ . The quantity  $N - 1$  is called the *branching order* of  $F$  at  $p$ .

### 2.2 The Grassmannian

We denote by  $G_2^+(\mathbb{R}^4)$  the Grassmannian of oriented 2-planes in  $\mathbb{R}^4$ . An oriented 2-plane  $P$  can be viewed as the 2-vector  $e_1 \wedge e_2$  where  $(e_1, e_2)$  is a positive orthonormal basis of  $P$ . Thus  $G_2^+(\mathbb{R}^4)$  is embedded in  $\Lambda^2(\mathbb{R}^4)$ ; if we write  $P$  as a 2-vector we can define

$$H = \frac{1}{\sqrt{2}}(P + \star P) \quad K = \frac{1}{\sqrt{2}}(P - \star P) \quad (2)$$

where  $\star : \Lambda^2(\mathbb{R}^4) \longrightarrow \Lambda^2(\mathbb{R}^4)$  is the Hodge operator ([Be] or [Jo] p. 82).

The 2-vector  $H$  (resp.  $K$ ) defined in (2) belongs to the unit sphere of  $\Lambda^+(\mathbb{R}^4)$  (resp.  $\Lambda^-(\mathbb{R}^4)$ ) and we derive an identification

$$G_2^+(\mathbb{R}^4) \cong \mathbb{S}(\Lambda^+(\mathbb{R}^4)) \times \mathbb{S}(\Lambda^-(\mathbb{R}^4)) \quad (3)$$

### 2.3 The Gauss map

If  $F : \mathbb{D} \longrightarrow \mathbb{R}^4$  is an immersion, we derive two Gauss maps

$$\gamma_+ : \mathbb{D} \longrightarrow \mathbb{S}(\Lambda^+(\mathbb{R}^4)), \quad \gamma_- : \mathbb{D} \longrightarrow \mathbb{S}(\Lambda^-(\mathbb{R}^4)) \quad (4)$$

as follows. We let  $z \in \mathbb{D}$  and let  $P$  be the oriented tangent plane  $F_*(T_z\mathbb{D})$ ; the orientation on  $P$  is defined via  $F$  by the orientation on  $\mathbb{D}$ . Using (2), we write  $P = \frac{1}{\sqrt{2}}(H + K)$  where  $H$  (resp.  $K$ ) of  $\Lambda^+(\mathbb{R}^4)$  (resp.  $\Lambda^-(\mathbb{R}^4)$ ). We let

$$H = \gamma_+(z) \quad K = \gamma_-(z) \quad (5)$$

Note that there is another way of defining the Gauss map: we write  $F$  in components as  $F = (F_1, F_2, F_3, F_4)$  and define for every  $i = 1, \dots, 4$  the complex number

$$\phi_i = \frac{\partial F_i}{\partial x} - i \frac{\partial F_i}{\partial y} \quad (6)$$

Identifying the 2-spheres  $\mathbb{S}(\Lambda^+(\mathbb{R}^4))$  and  $\mathbb{S}(\Lambda^-(\mathbb{R}^4))$  with the complex projective line  $\mathbb{C}P^1$ , we can write ([M-O])

$$\gamma_+ = \frac{\phi_3 + i\phi_4}{\phi_1 - i\phi_2} \quad \gamma_- = \frac{-\phi_3 + i\phi_4}{\phi_1 - i\phi_2} \quad (7)$$

## 2.4 The symplectic structures associated to the branch point

The tangent plane at  $p$  to  $F(\mathbb{D})$  in §2.1 is the plane  $P_0$  generated by  $\frac{\partial}{\partial x_1}$  and  $\frac{\partial}{\partial x_2}$ ; we orient it by taking  $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$  to be a positive basis; it is a complex line for two orthogonal complex structures on  $\mathbb{R}^4$ , one for each orientation. In terms of 2-vectors, these complex structures are written

$$H_0 = \frac{1}{\sqrt{2}}(P_0 + *P_0) \quad K_0 = \frac{1}{\sqrt{2}}(P_0 - *P_0) \quad (8)$$

The 2-vectors  $H_0$  and  $K_0$  define symplectic forms  $\omega_+$  and  $\omega_-$  on  $\mathbb{R}^4$  as follows

$$\omega_+(u, v) = \langle H_0, u \wedge v \rangle \quad \omega_-(u, v) = \langle K_0, u \wedge v \rangle \quad (9)$$

for two vectors  $u, v \in \mathbb{R}^4$  ( $\langle, \rangle$  denotes the scalar product on 2-vectors).

In a neighbourhood of  $p$ , a tangent plane  $P$  to  $F(\mathbb{D})$  is symplectic for both  $\omega_+$  and  $\omega_-$ , that is, it verifies

$$\langle P, H_0 \rangle > 0 \quad (10)$$

$$\langle P, K_0 \rangle > 0. \quad (11)$$

Unlike in the case of a complex curve in a complex surface, there is no preferred orientation associated to a minimal surface, so we consider both these symplectic structures.

## 2.5 The knot of the branch point

In this section we assume that the map  $F$  defined in 2.1 is a topological embedding.

Given a small positive number  $\epsilon$ , we denote by  $\mathbb{S}_\epsilon$  (resp.  $\mathbb{B}_\epsilon$ ) the sphere (resp. ball) centered at  $p$  and of radius  $\epsilon$ . If  $\epsilon$  is small enough,  $K^\epsilon = \mathbb{S}_\epsilon \cap F(D)$  is a knot and  $F(\mathbb{D} \cap \mathbb{B}_\epsilon)$  is homeomorphic to the cone on  $K^\epsilon$  (cf. [S-V1] where this construction follows from [Mi]).

## 2.6 The braid defined by the knot $K^\epsilon$ and its writhe number

The knot  $K^\epsilon$  is naturally presented as a braid with  $N$  strands in the 3-sphere (cf. [Vi]); the axis of this braid is the great circle in the normal plane at 0, that is the plane which is orthogonal to the tangent plane at 0 generated by  $\frac{\partial}{\partial x_1}$  and  $\frac{\partial}{\partial x_2}$ . The *algebraic crossing number* of this braid is

$$e(K^\epsilon) = lk(K^\epsilon, \hat{K}^\epsilon) \quad (12)$$

where  $\hat{K}^\epsilon$  is the knot obtained by pushing slightly  $K^\epsilon$  in the direction of the axis of the braid.

REMARK. In [S-V1], we consider the knot in the cylinder  $\{(z_1, z_2) \in \mathbb{C}^2 / |z_1| = \eta\}$ ; and in [S-V2] we use the terme *writhe* instead of *algebraic crossing number*.

## 3 Desingularization of a branch point

**Theorem 1.** *Let  $F : \mathbb{D} \longrightarrow \mathbb{R}^4$  be a minimal map with a branch point as in §2.1.*

*For some real number  $\epsilon > 0$  there exists, for  $t \in [0, \epsilon)$ , a smooth family  $F_t^{(+)} : \mathbb{D} \longrightarrow \mathbb{R}^4$  (resp.  $F_t^{(-)} : \mathbb{D} \longrightarrow \mathbb{R}^4$ ) of minimal immersions such that*

*1)  $F_0^{(-)} = F_0^{(+)} = F$ .*

*For every  $t$  small enough,*

*2)  $F_t$  is an immersion with transverse double points.*

*3)  $F_t^{(+)}$  (resp.  $F_t^{(-)}$ ) is symplectic w.r.t.  $\omega_+$  (resp.  $\omega_-$ ).*

*If  $F$  is an embedding, we have*

*4) The numbers  $D^{(+)}$ ,  $D^{(-)}$  of double points of  $F_t^{(+)}$ ,  $F_t^{(-)}$  verify*

$$2D^{(+)} = e(K) - (N - 1) \quad (13)$$

$$2D^{(-)} = -w(K) - (N - 1) \quad (14)$$

where  $N - 1$  is the branching order (cf. §2.1).

PROOF OF THEOREM 1.

Each coordinate function  $F_i$ ,  $i = 1, \dots, 4$  is harmonic, hence there exist four holomorphic functions  $f_1, \dots, f_4$  such that

$$F_1 + iF_2 = f_1 + \bar{f}_2 \quad F_3 + iF_4 = f_3 + \bar{f}_4 \quad (15)$$

Since  $F$  is a conformal map, the  $f_i$ 's verify (cf. [M-W])

$$f_1' f_2' + f_3' f_4' = 0 \quad (16)$$

**Lemma 1.** *A point  $z_0$  in  $\mathbb{D}$  is a branch point if and only if for every  $i = 1, \dots, 4$*

$$f_i'(z_0) = 0$$

*Proof.* The point  $z_0$  is a branch point if and only if  $\frac{\partial F}{\partial x}(z_0) = \frac{\partial F}{\partial y}(z_0) = 0$ . Lemma 1 follows from looking at the formulae for the derivatives of  $F$

$$\frac{\partial F}{\partial x} = \begin{pmatrix} \operatorname{Re}(f_1' + f_2') \\ \operatorname{Im}(f_1' - f_2') \\ \operatorname{Re}(f_3' + f_4') \\ \operatorname{Im}(f_3' - f_4') \end{pmatrix}$$

$$\frac{\partial F}{\partial y} = \begin{pmatrix} -\operatorname{Im}(f_1' + f_2') \\ \operatorname{Re}(f_1' - f_2') \\ -\operatorname{Im}(f_3' + f_4') \\ \operatorname{Re}(f_3' - f_4') \end{pmatrix}$$

□

We now assume  $z_0 = 0$  which causes no loss of generality.

Going back to the assumptions of Th. 1, we derive the existence of holomorphic functions  $\tilde{f}_i$ 's and positive integers  $n_i$ ,  $i = 1, \dots, 4$  such that for every  $i = 1, \dots, 4$

$$f_i' = z^{n_i} \tilde{f}_i \quad (17)$$

with  $\tilde{f}_i(0) \neq 0$ . We derive from that (16) that

$$n_1 + n_2 = n_3 + n_4 \quad (18)$$

Without loss of generality, we assume that  $n_1 < n_2, n_3, n_4$ . It follows from (18) that  $\tilde{f}_3$  and  $\tilde{f}_4$  have order smaller than  $\tilde{f}_2$ .

We construct the  $F_t^{(+)}$ 's and we indicate what to change to construct the  $F_t^{(-)}$ 's.

For  $A = (a_0, \dots, a_{n_1}) \in \mathbb{C}^{n_1+1}$  and  $B = (b_0, \dots, b_{n_3}) \in \mathbb{C}^{n_3+1}$ , we let

$$h_1(z, A, B) = (z^{n_1} + \sum_{i=0}^{n_1} a_i z^i) \tilde{f}_1(z) \quad h_2(z, A, B) = z^{n_2-n_3} (z^{n_3} + \sum_{i=0}^{n_3} b_i z^i) \tilde{f}_2(z) \quad (19)$$

$$h_3(z, A, B) = (z^{n_3} + \sum_{i=0}^{n_3} b_i z^i) \tilde{f}_3(z) \quad h_4(z, A, B) = z^{n_4-n_1} (z^{n_1} + \sum_{i=0}^{n_1} a_i z^i) \tilde{f}_4(z) \quad (20)$$

The  $h_i$ 's are holomorphic and verify (using (16) and (18))

$$h_1 h_2 + h_3 h_4 = 0 \quad (21)$$

For  $i = 1, \dots, 4$ , we let

$$f_i(z, A, B) = \int_0^z h_i(\xi, A, B) d\xi \quad (22)$$

The  $f_i(\cdot, A, B)$ 's are holomorphic and verify  $\frac{\partial f_i}{\partial z} = h_i$ . We let

$$F(z, A, B) = (f_1(z, A, B) + \bar{f}_2(z, A, B), f_3(z, A, B) + \bar{f}_4(z, A, B)).$$

It follows from (21) that for every  $(A, B)$ , the  $F(\cdot, A, B)$ 's are minimal maps. We assume that  $(A, B)$  belongs to the open dense set  $X_1$  of  $\mathbb{C}^{n_1+1} \times \mathbb{C}^{n_4+1}$  of the  $(A, B)$ 's such that the polynomials  $z^{n_1} + \sum_{i=0}^{n_1} a_i z^i$  and  $z^{n_4} + \sum_{i=0}^{n_4} b_i z^i$  have distinct roots which are all different from 0. It follows from Lemma 1 that for  $(A, B)$  in  $X_1$ ,  $F(\cdot, A, B)$  is an immersion.

We compute their Gauss maps using (7) and we see that

$$\gamma_+(F(\cdot, A, B)) = \frac{h_3(\cdot, A, B)}{h_2(\cdot, A, B)} = z^{n_3-n_2} \frac{\tilde{f}_3}{\tilde{f}_2} = \frac{f'_3}{f'_2} = \gamma_+(F) \quad (23)$$

It follows that the  $F(\cdot, A, B)$ 's are symplectic w.r.t.  $\omega_+$ .

Note that if we want the  $F(\cdot, A, B)$ 's to be symplectic w.r.t.  $\omega_-$ , we define instead

$$h_1(z, A, B) = (z^{n_1} + \sum_{i=0}^{n_1} a_i z^i) \tilde{f}_1(z) \quad h_2(z, A, B) = z^{n_2-n_4} (z^{n_4} + \sum_{i=0}^{n_4} b_i z^i) \tilde{f}_2(z) \quad (24)$$

$$h_3(z, A, B) = z^{n_3-n_1}(z^{n_1} + \sum_{i=0}^{n_1} a_i z^i) \tilde{f}_3(z) \quad h_4(z, A, B) = (z^{n_4} + \sum_{i=0}^{n_4} b_i z^i) \tilde{f}_4(z) \quad (25)$$

We will now use the Transversality Lemma to prove that for generic  $A, B$ ,  $F(\cdot, A, B)$  has only transverse double points. We do it for the functions defined in (19) and (20); the proof for (24) and (25) works identically.

We define

$$\begin{aligned} \Phi : \mathbb{C}^{n_1+1} \times \mathbb{C}^{n_3+1} \times \mathbb{D} \times \mathbb{D} &\longrightarrow \mathbb{R}^4 \times \mathbb{R}^4 \\ (A, B, z_1, z_2) &\mapsto (F(z_1, A, B), F(z_2, A, B)) \end{aligned}$$

and we prove

**Lemma 2.** *There exists a positive number  $\eta$  such that, for every  $A, B$ , if  $z_1 \neq z_2$  and  $|z_1| < \eta$ ,  $|z_2| < \eta$ , then  $\Phi$  is transverse to the diagonal  $\Delta$  of  $\mathbb{R}^4 \times \mathbb{R}^4$  at  $(A, B, z_1, z_2)$ .*

*Proof.* We identify  $\mathbb{R}^4$  with  $\mathbb{C}^2$ ; if  $J_0$  is the canonical complex structure on  $\mathbb{C}^2$ , we introduce a new orthogonal complex structure  $J_1$  on  $\mathbb{C}^2$  defined

$$J_1(1, 0) = (i, 0) \quad J_1(0, 1) = (0, -i) \quad (26)$$

The point of this change is to make  $F$  holomorphic w.r.t.  $A$  and antiholomorphic w.r.t.  $B$ . If we use (24) and (25), the map  $F$  is holomorphic in  $A$  and antiholomorphic in  $B$  for the standard complex structure  $J_0$  so we keep it.

The diagonal  $\Delta$  is a complex subspace of  $\mathbb{C}^4$  which is generated over the complex numbers by the vectors

$$\epsilon_1 = (1, 0, 1, 0) \quad \epsilon_2 = (0, 1, 0, 1) \quad (27)$$

If  $i = 0, \dots, n_1$  (resp.  $j = 0, \dots, n_3$ ), we write  $a_i$  (resp.  $b_j$ ) in real coordinates

$$a_i = a_i^{(1)} + ia_i^{(2)} \quad (\text{resp.} \quad b_j = b_j^{(1)} + ib_j^{(2)}) \quad (28)$$

The map  $F$  is now holomorphic in  $A$  and antiholomorphic in  $B$ , hence Lemma 2 will be proved once we prove

**Lemma 3.**

$$\det\left(\frac{\partial \Phi}{\partial a_0}, \overline{\frac{\partial \Phi}{\partial b_0}}, \epsilon_1, \epsilon_2\right) \neq 0$$

*the determinant being computed over the complex numbers.*

*Proof.* We have

$$\frac{\partial \Phi}{\partial a_0}(A, B, z_1, z_2) = \left( \frac{\partial F}{\partial a_0}(z_1, A, B), \frac{\partial F}{\partial a_0}(z_2, A, B) \right) \in \mathbb{C}^2 \times \mathbb{C}^2 \quad (29)$$

For  $i = 1, 2$ , we write in Euclidean complex coordinates in  $\mathbb{C}^2$ ,

$$\begin{aligned} \frac{\partial F}{\partial a_0}(z_i, A, B) &= \left( \frac{\partial f_1}{\partial a_0}(z_i, A, B), \frac{\partial f_4}{\partial a_0}(z_i, A, B) \right) \\ &= \left( \int_0^{z_i} \frac{\partial h_1}{\partial a_0}(\xi, A, B) d\xi, \int_0^{z_i} \frac{\partial h_4}{\partial a_0}(\xi, A, B) d\xi \right) \in \mathbb{C}^2 \end{aligned} \quad (30)$$

by differentiation under the integral sign, hence

$$\begin{aligned} \frac{\partial \Phi}{\partial a_0}(A, B, z_1, z_2) &= \\ &= \left( \int_0^{z_1} \frac{\partial h_1}{\partial a_0}(\xi, A, B) d\xi, \int_0^{z_1} \frac{\partial h_4}{\partial a_0}(\xi, A, B) d\xi, \int_0^{z_2} \frac{\partial h_1}{\partial a_0}(\xi, A, B) d\xi, \int_0^{z_2} \frac{\partial h_4}{\partial a_0}(\xi, A, B) d\xi \right) \end{aligned} \quad (31)$$

$$\begin{aligned} \text{Similarly } \frac{\partial \Phi}{\partial \bar{b}_0}(A, B, z_1, z_2) &= \\ &= \left( \int_0^{z_1} \frac{\partial \bar{h}_2}{\partial \bar{b}_0}(\xi, A, B) d\xi, \int_0^{z_1} \frac{\partial \bar{h}_3}{\partial \bar{b}_0}(\xi, A, B) d\xi, \int_0^{z_2} \frac{\partial \bar{h}_2}{\partial \bar{b}_0}(\xi, A, B) d\xi, \int_0^{z_2} \frac{\partial \bar{h}_3}{\partial \bar{b}_0}(\xi, A, B) d\xi \right) \end{aligned} \quad (32)$$

We can now compute

$$\begin{aligned} \det\left(\frac{\partial \Phi}{\partial a_0}, \frac{\partial \Phi}{\partial \bar{b}_0}, \epsilon_1, \epsilon_2\right) &= \\ &= \int_{z_2}^{z_1} \frac{\partial h_1}{\partial a_0}(\xi, A, B) d\xi \int_{z_2}^{z_1} \frac{\partial \bar{h}_3}{\partial \bar{b}_0}(\xi, A, B) d\xi - \int_{z_2}^{z_1} \frac{\partial h_4}{\partial a_0}(\xi, A, B) d\xi \int_{z_2}^{z_1} \frac{\partial \bar{h}_2}{\partial \bar{b}_0}(\xi, A, B) d\xi \end{aligned} \quad (33)$$

We now compute the derivatives involved:

$$\frac{\partial h_1}{\partial a_0}(z, A, B) = \tilde{f}_1(z) \quad \frac{\partial h_2}{\partial b_0}(z, A, B) = z^{n_2-n_3} \tilde{f}_2(z) \quad (34)$$

$$\frac{\partial h_3}{\partial b_0}(z, A, B) = \tilde{f}_3(z) \quad \frac{\partial h_4}{\partial b_0}(z, A, B) = z^{n_4-n_1} \tilde{f}_4(z) \quad (35)$$



We remind the reader that  $\tilde{f}_1(0) \neq 0$  and  $\tilde{f}_3(0) \neq 0$ ; and on the other hand,  $n_2 - n_3 > 0$  and  $n_4 - n_1 > 0$ . This enables us to derive the existence of a positive constant  $C$  and of an  $\eta > 0$  such that, if  $|z_i| < \eta$ , for  $i = 1, 2$ , then

$$|\det(\frac{\partial \Phi}{\partial a_0}, \frac{\partial \Phi}{\partial b_0}, \epsilon_1, \epsilon_2)| = |(33)| \geq C|z_1 - z_2|^2$$

This concludes the proof of Lemmas 3 and 2.  $\square$

We derive from Lemma 2 and from the Transversality Lemma ([G-P]) the existence of a dense subset  $X_2$  of the product of the unit balls  $\mathbb{B}^{n_1+1} \times \mathbb{B}^{n_3+1}$  such that, if  $(A, B) \in X_2$ , the map

$$\Phi(A, B, \cdot, \cdot) : \{(z_1, z_2) \in \mathbb{D} \times \mathbb{D} / z_1 \neq z_2\} \longrightarrow \mathbb{R}^4 \times \mathbb{R}^4$$

is transversal to  $\Delta$ . It follows that, if  $(A, B) \in X_1 \cap X_2$ ,  $F(\cdot, A, B)$  has only transverse double points. To conclude the proof of Th. 1, we use the Curve Selection Lemma for subanalytic sets (see [B-M],[Lo]). In order to do this, we prove

**Lemma 4.**  *$X_1 \cap X_2$  is subanalytic.*

*Proof.* The complement of  $X_1$  is algebraic so  $X_1$  is semialgebraic, hence we just need to show that  $X_2$  is subanalytic. We let

$$\mathcal{Z} = \{(z_1, z_2, A, B) \in \mathbb{D} \times \mathbb{D} \times \mathbb{B}^{n_1+1} \times \mathbb{B}^{n_3+1} / \|A\| \leq 1, \|B\| \leq 1, z_1 \neq z_2 \text{ and}$$

$$\det(\frac{\partial F}{\partial x_1}(z_1, A, B), \frac{\partial F}{\partial y_1}(z_1, A, B), \frac{\partial F}{\partial x_2}(z_2, A, B), \frac{\partial F}{\partial y_2}(z_2, A, B), \epsilon_1, J_1 \epsilon_1, \epsilon_2, J_1 \epsilon_2)^2 \\ + \|F(z_1, A, B) - F(z_2, A, B)\|^2 = 0\}.$$

Note that here we are talking of the real determinant in  $\mathbb{R}^8$ ; it follows from its definition that  $\mathcal{Z}$  is semianalytic.

We let

$$\Pi : \mathbb{D} \times \mathbb{D} \times \mathbb{C}^{n_1+1} \times \mathbb{C}^{n_3+1} \longrightarrow \mathbb{C}^{n_1+1} \times \mathbb{C}^{n_3+1}$$

be the projection. Since  $\mathcal{Z}$  is semianalytic, the set  $\Pi(\mathcal{Z})$  is subanalytic. It follows that  $X_2 = \mathbb{B}^{n_1+1} \times \mathbb{B}^{n_3+1} \setminus \Pi(\mathcal{Z})$  is subanalytic (cf. the theorem of the complement, [B-M]).  $\square$

Since  $X_1 \cap X_2$  is subanalytic and dense, the Curve Selection Lemma ensures the existence of an analytic path

$$\gamma : [0, \epsilon) \longrightarrow X_1 \cap X_2$$

such that  $\gamma(0) = 0$  and for every  $t > 0$ ,  $\gamma(t) \in X_1 \cap X_2$ . We let  $F_t^{(+)}(z) = F(z, A(\gamma(t)), B(\gamma(t)))$ . If  $t > 0$ ,  $F_t^{(+)}$  is a minimal immersion with only transverse double points. This proves Th. 1 1), 2) and 3).

**Lemma 5.**  $\exists \eta_0$  such that  $\forall \eta < \eta_0, \exists t(\eta)$  such that  $\forall t, 0 < t < t(\eta)$ , the knots  $K_t^\eta = F_t^{(+)}(\mathbb{D}) \cap \mathbb{S}_\eta$  are transversally isotopic to  $K^\eta = F(\mathbb{D}) \cap \mathbb{S}_\eta$ .

*Proof.* There is a constant  $C$  such that, for  $t$  small enough,  $|A| \leq C|t|$  and  $|B| \leq C|t|$ .

Also, there exists an  $\eta_1$  such that, if  $\eta < \eta_1$  and  $|F(z)| = \eta$ , then

$$|\rho^N - \eta| \leq \frac{\eta}{10}.$$

So for  $\eta < \eta_1$ , we let  $t(\eta)$  such that, if  $t < t(\eta)$  and  $|F(z)| = \eta$ , then

$$|\rho^N - \eta| \leq \frac{\eta}{5} \tag{36}$$

We let  $z = \rho e^{i\theta} \in \mathbb{D}$ . We can derive from the construction of  $F_t^{(+)}$ , the following estimate

$$F_t^{(+)}(z) = \rho^N e^{Ni\theta} X + o(\rho^N) + \mathcal{O}(t) \tag{37}$$

where  $X = (1, 0, 0, 0) \in \mathbb{R}^4$ .

We need to say a word of what we mean by the  $\mathcal{O}(t)$ 's in this paragraph: these terms can contain terms in  $\rho^k$ , for  $k > 0$  (and/or later in the proof terms in  $\rho^{-k}$ ). So once  $\eta$  is fixed, we can derive  $t$  in terms of  $\eta$  (hence the notation  $t(\eta)$  in the statement of the lemma) such that the  $\mathcal{O}(t)$  is as small as we want. The term  $o(\rho^N)$  on the other hand, is independent of  $t$ .

The vectors  $\frac{1}{N}\rho\frac{\partial}{\partial\rho}$  and  $\frac{1}{N}\frac{\partial}{\partial\theta}$  are orthogonal and of the same norm in  $\mathbb{D}$ ; since the  $F_t^{(+)}$ 's are minimal, the vectors

$$u_1 = \frac{1}{N}\rho\frac{\partial F_t^{(+)}}{\partial\rho} \quad u_2 = \frac{1}{N}\frac{\partial F_t^{(+)}}{\partial\theta} \tag{38}$$

are orthogonal and of the same norm and they generate the plane tangent to  $F_t^{(+)}(\mathbb{D})$ . We have

$$u_1 = \rho^N e^{Ni\theta} X + o(\rho^N) + \mathcal{O}(t) \quad u_2 = \rho^N e^{Ni\theta} iX + o(\rho^N) + \mathcal{O}(t) \quad (39)$$

The vector  $\gamma$  tangent to  $K_t^\eta$  at  $F_t^{(+)}(z)$  is of the form  $\gamma = au_1 + bu_2$  and verifies

$$\langle F_t^{(+)}(z), \gamma \rangle = 0 \quad (40)$$

We derive from (40), (37) and (39) that

$$\langle \gamma, u_1 \rangle = \|\gamma\| (o(\rho^{2N}) + \mathcal{O}(t)) \quad (41)$$

Hence (remember that  $\|u_1\| = \|u_2\|$ )

$$\langle \gamma, u_2 \rangle^2 = \|\gamma\|^2 \langle u_1, u_1 \rangle^2 - \langle \gamma, u_1 \rangle^2 = \|\gamma\|^2 (\rho^{2N} + o(\rho^{2N}) + \mathcal{O}(t))$$

On the other hand,  $iF_t^{(+)}(z) - u_2 = o(\rho^N) + \mathcal{O}(t)$  hence

$$\begin{aligned} \langle \gamma, iF_t^{(+)}(z) \rangle^2 &= \langle \gamma, u_2 \rangle^2 + \|\gamma\|^2 (o(\rho^{2N}) + \mathcal{O}(t)) = \|\gamma\|^2 (\rho^{2N} + o(\rho^{2N}) + \mathcal{O}(t)) \\ &\geq \|\gamma\|^2 \left( \frac{\eta^2}{2} + o(\eta^2) + \mathcal{O}(t) \right) \end{aligned}$$

This last estimate is derived from (36). We derive the existence of a  $\eta_0 < \eta_1$  such that,  $\forall \eta < \eta_0$ ,  $\exists t(\eta)$  such that, if  $t < t(\eta)$ ,  $\langle \gamma, iF_t^{(+)}(z) \rangle \neq 0$ . We conclude that all the  $K_t^\eta$ 's are all transverse and they are all transversally isotopic.  $\square$

Note that the  $K_t^{(\eta)}$ 's are transverse w.r.t. the contact structures associated to *both* the symplectic structures. By contrast, the disks  $F_t^{(+)}(\mathbb{D})$  and  $F_t^{(-)}(\mathbb{D})$  are not symplectic for both structures.

The number  $D_t$  of transverse double points  $F_t^{(+)}$  is given by ([H-H])

$$2D_t = sl(K) + 1 = e(K) - (N - 1) \quad (42)$$

$\square$

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